

DIRECT PRODUCTS OF FREE GROUPS AND FREE IDEMPOTENT GENERATED SEMIGROUPS OVER BANDS

IGOR DOLINKA

ABSTRACT. For each group G which decomposes into a finitary direct product of free groups of finite rank we construct a regular band B such that the free idempotent generated semigroup over B contains a maximal subgroup isomorphic to G . In particular, there exists a (regular) band B_0 with the property that any idempotent generated semigroup whose biordered set is isomorphic to that of B_0 must have all its subgroups abelian.

1. INTRODUCTION

Let S be a semigroup. The set $E = E(S)$ of all idempotents of S carries a structure of a partial algebra, called the *biordered set* of S , by retaining the products of the so-called *basic pairs*: these are pairs of idempotents $\{e, f\}$ such that either $ef \in \{e, f\}$ or $fe \in \{e, f\}$ (note that if $ef \in \{e, f\}$ then fe is also an idempotent and the same is true if we interchange the roles of e and f). Therefore, if S is an idempotent semigroup (i.e. a *band*) then its biordered set [10] is in general different from S itself, since not every pair is necessarily basic. The term ‘biordered set’ comes from an alternative approach, when one considers a relational structure over $E(S)$ equipped with two partial orders related to basic pairs; here we shall not pursue this approach, directing instead to [11, 12, 18, 23] for further background.

The class of idempotent generated semigroups is of prime importance in semigroup theory, and it includes a host of natural examples: let us only mention the semigroup of singular (non-bijective) self-maps of a finite set (Howie [19]) and the semigroup of all singular $n \times n$ matrices over a field (Erdos [14]). It is not difficult to show that the category of all idempotent generated semigroups with a fixed biordered set E has an initial object $\mathbf{IG}(E)$, called the *free idempotent generated semigroup* over E (we shall also say ‘over S ’ when $E = E(S)$). This semigroup is defined by the presentation

$$\mathbf{IG}(E) = \langle E \mid e \cdot f = ef \text{ such that } \{e, f\} \text{ is a basic pair} \rangle,$$

where $e \cdot f$ is a word of length 2 in the free semigroup E^+ , while ef is an element of E . It has a fundamental part in understanding the structure of idempotent generated semigroups with a prescribed biordered set of idempotents.

For reasons that are intrinsic to semigroup theory [18, 20], much of that structure depends upon the knowledge of maximal subgroups of $\mathbf{IG}(E)$. It was conjectured for a long time that for any biordered set E , each maximal subgroup of $\mathbf{IG}(E)$ is free; although this conjecture was widely circulated back in the eighties of the last century, it was explicitly recorded only in [21]. Indeed, this was proved to be true for a number of particular cases, see e.g. [21, 24, 26]. However, in 2009, Brittenham, Margolis and Meakin [3] came up with an example of a 72-element semigroup S such that $\mathbf{IG}(E(S))$ has a maximal subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, the free abelian group of rank 2 (and the fundamental group of the torus). But the biggest breakthrough and surprise came with the seminal paper of Gray and Ruškuc [16] who, using the Reidemeister-Schreier rewriting technique for subgroups

2010 *Mathematics Subject Classification.* 20M05, 20F05.

Key words and phrases. free idempotent generated semigroup, maximal subgroup, direct product, free group.

of semigroups and monoids developed earlier by Ruškuc [25], proved that contrary to the previous conjecture *every* group arises as a maximal subgroup of $\text{IG}(E(S))$ for a suitably chosen semigroup S ; if the group in question is finitely presented then a finite S will suffice. This approach was subsequently exploited by Gray and Ruškuc [17] and Dolinka [7] who proved that symmetric groups arise as maximal subgroups of $\text{IG}(E(\mathcal{T}_n))$ and $\text{IG}(E(\mathcal{PT}_n))$, where \mathcal{T}_n and \mathcal{PT}_n are the monoids of all transformations and of all partial transformations of an n -element set, respectively. Furthermore, a recent contribution by Dolinka and Gray [8] proves that if $M_n(Q)$ denotes the full $n \times n$ matrix monoid over a skew field Q , then the maximal subgroups of $\text{IG}(E(M_n(Q)))$ corresponding to \mathcal{D} -classes of matrices of rank $r < n/3$ are precisely general linear groups $\text{GL}_r(Q)$.

In a slightly different offshoot of this developing theory, the present author initiated in [6] the investigation into maximal subgroups of free idempotent generated semigroups over (biordered set of) bands. (In the following, we are going to abuse the notation slightly and write $\text{IG}(B)$ instead of $\text{IG}(E)$, where E is the biordered set of a band B .) As a generalisation of results of Pastijn [26], it was proved in [6] that for a variety of bands \mathbf{V} we have that all maximal subgroups of $\text{IG}(B)$ are free for each $B \in \mathbf{V}$ if and only if \mathbf{V} is contained in one of the varieties **LSNB** and **RSNB** of left seminormal bands and right seminormal bands, respectively (see [27]). To prove this, one of the main steps was to exhibit a concrete example of a *regular band* B , an idempotent semigroup satisfying the identical law $xyxzx = xyzx$, such that $\text{IG}(B)$ contains a non-free maximal subgroup. It turned out that a certain 20-element subband B of the rank 4 free object of the variety **RB** of regular bands (of which **LSNB** and **RSNB** are the only maximal subvarieties) has a maximal subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. The band B consists of two \mathcal{D} -classes, and in [6, Figure 2] it becomes apparent that the (algebraic) action of the ‘upper’ \mathcal{D} -class on the ‘lower’ one is precisely (topologically) equivalent to the folding of a square into a torus. Drawing an inspiration from this example, it is the aim of the present note to prove the following result.

Theorem 1. *Let n, r_1, \dots, r_n be positive integers. For each finitary direct product of free groups of finite rank*

$$G = \prod_{i=1}^n F_{r_i}$$

there exists a (finite) regular band B such that $\text{IG}(B)$ contains a maximal subgroup isomorphic to G .

Direct products of free groups appear in a variety of contexts in group theory, with an abundance of applications. For example, they have a surprisingly rich and involved subgroup structure [1, 2, 5, 22, 28], and arise as fundamental groups of complements of certain line arrangements in the complex plane [13, 15, 29]. Since to each biordered set E of a semigroup corresponds a complex $GH(E)$, called the *Graham-Houghton complex* [3], such that the maximal subgroups of $\text{IG}(E)$ coincide with fundamental groups of $GH(E)$ at its various points, in this note we are going to obtain yet another realisation of direct products of free groups as fundamental groups of dimension 2 cell complexes. In particular, we are going to single out an example of a regular band whose biordered set E has the property that for any idempotent generated semigroup S such that $E(S)$ is isomorphic to E any subgroup of S must be abelian.

The remainder of the paper is organised as follows. The necessary notions, definitions and known results are collected in the next section; this includes recalling from [6] the presentation of a maximal subgroup of $\text{IG}(B)$ for a band B . The construction of a regular band B required by Theorem 1 and its main properties will be presented in Sect. 3. Finally, the proof of Theorem 1 occupies Sect. 4, along with related discussion.

2. PRELIMINARIES

We refer to [18, 20, 27] for a general background in semigroup theory; still, we recall some of the most important basic notions. First of all, one of the fundamental motifs in studying the structure of a semigroup is classifying its elements according to the (left, right, two-sided) principal ideals they generate. If S^1 denotes the monoid obtained from a semigroup S by adjoining an identity element, we define the following *Green's relations*:

$$a\mathcal{L}b \Leftrightarrow S^1a = S^1b, \quad a\mathcal{R}b \Leftrightarrow aS^1 = bS^1, \quad a\mathcal{J}b \Leftrightarrow S^1aS^1 = S^1bS^1.$$

In addition, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, while \mathcal{D} is the least equivalence on S containing both \mathcal{L} and \mathcal{R} . For periodic (in particular, for finite) semigroups we always have $\mathcal{D} = \mathcal{J}$. A \mathcal{D} -class D of S is *regular* if all elements of D are (von Neumann) regular; an equivalent statement is that D contains an idempotent. \mathcal{H} -classes that contain an idempotent are precisely the maximal subgroups of S . All such subgroups contained within the same \mathcal{D} -class are isomorphic.

Therefore, the relation \mathcal{H} is trivial on any band B , as each \mathcal{H} -class is a trivial subgroup of B . So, each \mathcal{D} -class D of B can be thought of as a rectangular array of elements, where each column represents an \mathcal{L} -class, while each row is an \mathcal{R} -class; if $a, b \in D$ then the product ab is the unique entry lying in the same row as a and in the same column as b (hence D is a *rectangular band*). In fact, the relation \mathcal{D} is always a congruence of B and the quotient B/\mathcal{D} is a semilattice called the *structure semilattice* of B .

If E is the biordered set of an idempotent generated semigroup S , then by the very definition of $\text{IG}(E)$ there is a natural homomorphism $\phi : \text{IG}(E) \rightarrow S$ such that its restriction to E is an isomorphism of biordered sets [12, 23]. Furthermore, ϕ induces a bijection between the (regular) \mathcal{D} -classes of $\text{IG}(E)$ and S , and for each idempotent $e \in E$ it maps the \mathcal{R} -class R_e (resp. the \mathcal{L} -class L_e) in $\text{IG}(E)$ containing e onto the corresponding \mathcal{R} -class (resp. \mathcal{L} -class) of S . Consequently, the restriction of ϕ to the maximal subgroup H_e of $\text{IG}(E)$ containing e is a group homomorphism onto the maximal subgroup $H'_e \ni e$ of S .

If B is a band and $e \in B$, it is precisely this maximal subgroup H_e of $\text{IG}(B)$ that we aim to compute. There is no loss of generality in assuming that the D_e , the \mathcal{D} -class of B containing e , is in fact the (unique) minimal \mathcal{D} -class of B : otherwise, we may restrict our attention only to those \mathcal{D} -classes of B which are greater than or equal to D_e in the induced semilattice order of B/\mathcal{D} , thus obtaining a subband B' of B . The main general result of [16], Theorem 5, shows that \mathcal{D} -classes D such that $D_e \not\leq D$ do not influence the presentation of H_e in any way, so that the maximal subgroup of B' containing e will be absolutely the same as that of B . However, we shall not invoke here this general result, as it would require additional technical preparations and notions; instead, we focus on the particular case of biordered sets of bands, analysed in detail in [6].

To this end, with all the previous conventions in mind, we shall represent the elements of D_e by pairs from the direct product $I \times J$, where the index sets I and J share a common element 0; the pair $(0, 0)$ (in the top left corner) will be identified with e . Now each element a of B induces a pair of transformations (λ_a, ρ_a) on I and J , respectively (the former will be written to the left from its argument, and the latter to the right), defined by the rules:

$$a(i, j) = (\lambda_a(i), j), \quad (i, j)a = (i, (j)\rho_a)$$

for all $i \in I$ and $j \in J$. We are finally in position to recall (in our notation) the result describing the presentation of the maximal subgroup of $\text{IG}(B)$ containing e .

Theorem 2 (Corollary 5 of [6]). *Let B be a band and $e \in B$. Assume that the set I indexes the \mathcal{R} -classes contained in D_e , while J does the same for \mathcal{L} -classes of D_e . Let $0 \in I \cap J$ and assume that e is represented by the coordinates $(0, 0)$. The maximal subgroup H_e of $\text{IG}(B)$ containing e is defined by the presentation $\langle \mathfrak{X} | \mathfrak{R} \rangle$ where $\mathfrak{X} = \{f_{ij} : i \in I, j \in J\}$, while \mathfrak{R} consists of the relations*

$$f_{i0} = f_{0j} = 1 \tag{2.1}$$

for all $i \in I$, $j \in J$, and

$$f_{ij}^{-1} f_{i\ell} = f_{kj}^{-1} f_{k\ell}, \quad (2.2)$$

where for some $a \in B$ such that $D_e \leq D_a$ the indices $i, k \in I$, $j, \ell \in J$ satisfy one of the following two conditions:

- (a) $\lambda_a(i) = i$, $\lambda_a(k) = k$, $(j)\rho_a = (\ell)\rho_a = \ell$;
- (b) $\lambda_a(i) = \lambda_a(k) = k$, $(j)\rho_a = j$, $(\ell)\rho_a = \ell$.

In the case when one of the two latter sets of conditions are satisfied we say that the element a *singularises* the ‘square’ $(i, k; j, \ell)$. If (a) holds we say that the singularisation is of the *left-right* type, while in case (b) it is of the *up-down* type. An equivalent, more compact way of expressing these conditions (that will be of use later) is: (a) $i, k \in \text{Im } \lambda_a$, $\ell \in \text{Im } \rho_a$, $(j, \ell) \in \text{Ker } \rho_a$, and (b) $k \in \text{Im } \lambda_a$, $(i, k) \in \text{Ker } \lambda_a$, $j, \ell \in \text{Im } \rho_a$. This follows immediately from the fact that both λ_a and ρ_a are idempotent transformations of their respective sets. In fact, these conditions may be further simplified. It is this simplified version, described by the following lemma, that will be used throughout the paper, and the terms ‘left-right singular’ and ‘up-down singular’ will also include the squares satisfying the conditions (a') and (b') below.

Lemma 3. *Theorem 2 holds true if the conditions (a) and (b) are replaced by the following ones:*

- (a') $i, k \in \text{Im } \lambda_a$, $(j, \ell) \in \text{Ker } \rho_a$;
- (b') $(i, k) \in \text{Ker } \lambda_a$, $j, \ell \in \text{Im } \rho_a$.

Proof. Assume first that $(i, k; j, \ell)$ is a square satisfying (a'). Then $(j)\rho_a = (\ell)\rho_a = j'$ and so $(j')\rho_a = j'$, since ρ_a is idempotent. Consequently, both squares $(i, k; j, j')$ and $(i, k; \ell, j')$ are left-right singularised by a , thus the following relations are part of the presentation for H_e given in Theorem 2:

$$f_{ij}^{-1} f_{ij'} = f_{kj}^{-1} f_{kj'} \quad \text{and} \quad f_{i\ell}^{-1} f_{ij'} = f_{k\ell}^{-1} f_{kj'}.$$

By multiplying the first relation by the inverse of the second, we get:

$$f_{ij}^{-1} f_{i\ell} = (f_{ij}^{-1} f_{ij'}) (f_{i\ell}^{-1} f_{ij'})^{-1} = (f_{kj}^{-1} f_{kj'}) (f_{k\ell}^{-1} f_{kj'})^{-1} = f_{kj}^{-1} f_{k\ell}.$$

Hence, the relation (2.2) corresponding to the square $(i, k; j, \ell)$ may be added to the presentation without changing the presented group. The condition (b') is handled similarly. \square

Since we shall be dealing with regular bands, is it appropriate here to record their important feature.

Lemma 4. *Let B a regular band, and let $a, b, e \in B$ such that $I \times J = D_e \leq D_a = D_b$. Then:*

- (i) *if $a \mathcal{R} b$ then $\lambda_a = \lambda_b$ and $\text{Ker } \rho_a = \text{Ker } \rho_b$;*
- (ii) *if $a \mathcal{L} b$ then $\rho_a = \rho_b$ and $\text{Ker } \lambda_a = \text{Ker } \lambda_b$.*

Proof. Let $c = (i, j) \in I \times J$ be arbitrary. The condition $a \mathcal{R} b$ implies that $ax = b$ and $by = a$ for some $x, y \in B^1$; this is clearly equivalent to $ab = b$ and $ba = a$. Also, since $cac, cbc \in D_c = D_e$ we have $cac = c(cac)c = c$ and similarly $cbc = c$ and $bab = b$. By applying the identity $xyxzx = xyzx$ and the equalities just deduced, we obtain:

$$ac = bac = (bacb)ac = babc(ba)c = (bab)(cac) = bc,$$

so that $\lambda_a = \lambda_b$.

Furthermore, let $(j, \ell) \in \text{Ker } \rho_a$. Then $(j)\rho_a = (\ell)\rho_a$, implying $(j)\rho_b = (j)\rho_a\rho_b = (\ell)\rho_a\rho_b = (\ell)\rho_b$, so $(j, \ell) \in \text{Ker } \rho_b$. By exchanging the roles of a and b , we obtain the converse implication.

The statement (ii) follows analogously. \square

Remark 5. Actually, the converse of the previous lemma holds as well in the following sense: if B is a band such that the first parts of the conditions (i) and (ii) hold for any pair of comparable \mathcal{D} -classes $D_1 \leq D_2 \ni a, b$, then B must be regular, cf. [27, Proposition II.3.6].

In conclusion, if D is a fixed \mathcal{D} -class of B such that $D \geq D_e$, then any element $a \in D$ induces the same pair of equivalences $\text{Ker } \lambda_a, \text{Ker } \rho_a$ on I and J , respectively; in other words, to each \mathcal{D} -class above D_e we can associate a fixed partition π of I and a partition θ of J . Keeping track of these partitions will be essential in making our construction work. Also, notice that $\text{Im } \lambda_a$ ($\text{Im } \rho_a$) depends only on the \mathcal{L} -class (resp. \mathcal{R} -class) of a , and that it is always a cross-section of π (resp. θ). Therefore, bearing in mind Lemma 3, the singular squares in the grid $I \times J$ can be ‘clustered’ into what we call *singularisation zones*. To explain this concept visually, we depict typical zones in Fig. 1.

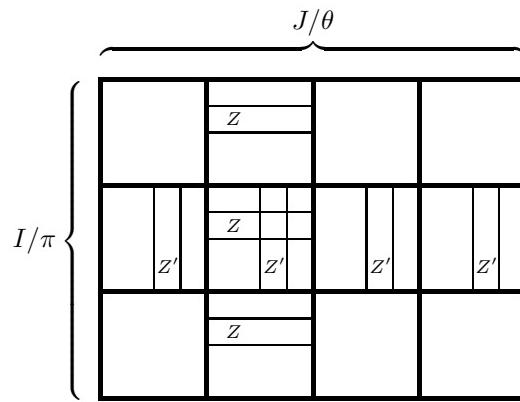


FIGURE 1. Left-right and up-down singularisation zones in a \mathcal{D} -class of a regular band

Here each rectangle labelled by Z represents a piece of a row contained in a fixed class from J/θ , and the rows in question are indexed by the elements of $\text{Im } \lambda_a$ for some a . The union of these pieces has the property that each square contained in it is left-right singularised (in the broader sense, suggested by Lemma 3) by some element $a \in B$ (or any other element of the \mathcal{R} -class R_a). So, this is a *left-right singularisation zone* induced by a . Dually, the union of pieces of columns labelled by Z' (indexed by elements of $\text{Im } \rho_b$ for some b) is the *up-down singularisation zone* induced by b , with an analogous property as above.

3. THE CONSTRUCTION

Given a sequence r_1, \dots, r_n of ranks of free groups the direct product of which is G , we define a band $B(r_1, \dots, r_n)$ as follows. Its elements will be pairs of s -tuples, $1 \leq s \leq n$, written

$$(p_1, \dots, p_s; q_s, \dots, q_1)$$

(for notational convenience we enumerate the second tuple backwards), where each $p_m \in \{0, 1\}$ and $0 \leq q_m \leq r_m$ for all $1 \leq m \leq s$. We shall make use of a more compact notation such that for a sequence $u \in \{p, q\}$ and $1 \leq m, m' \leq s$, $u[m, m']$ is a short-hand for $(u_m, \dots, u_{m'})$. Now the multiplication of these tuples is defined by

$$(p[1, s]; q[s, 1])(p'[1, t]; q'[t, 1]) = \begin{cases} (p[1, s], p'[s+1, t]; q'[t, 1]) & \text{if } s \leq t, \\ (p[1, s]; q[s, t+1], q'[t, 1]) & \text{if } s > t. \end{cases}$$

Lemma 6. $B(r_1, \dots, r_n)$ is a regular band. Its \mathcal{D} -classes form a chain $D_1 > \dots > D_n$, where for each $1 \leq s \leq n$, D_s consists of all pairs $(p[1, s]; q[s, 1])$ of sequences of length s .

Proof. As we have already seen in the proof of Lemma 4, $(p; q) \mathcal{R}(p'; q')$ holds if and only if $(p; q)(p'; q') = (p'; q')$ and $(p'; q')(p; q) = (p; q)$. By the definition of the multiplication in $B(r_1, \dots, r_n)$, all sequences p, p', q, q' must have the same length; the same conclusion follows from the assumption $(p; q) \mathcal{L}(p'; q')$, and thus from $(p; q) \mathcal{D}(p'; q')$. Conversely, assume that $(p[1, s], q[s, 1]), (p'[1, s], q'[s, 1]) \in B(r_1, \dots, r_n)$. By definition, if u, v, u', v' are all of the same length, then $(u; v)(u'; v') = (u; v')$. Therefore, we conclude that $(p; q) \mathcal{R}(p'; q') \mathcal{L}(p'; q')$ holds, and so $(p; q) \mathcal{D}(p'; q')$.

The fact that $B(r_1, \dots, r_n)$ is a band is obvious. The claim that it is regular may be verified by performing the routine task of checking the identity $xyxzx = xyzx$ while discussing all possible relations between lengths of pairs of sequences substituted for x, y, z . However, instead of that, we may also apply the criterion from Remark 5; thus let $s < t$, $(p; q), (p'; q') \in D_s$ and $(u; v) \in D_t$. If $(p; q) \mathcal{R}(p'; q')$ then, as we have basically already seen, $p = p'$, which implies

$$(p; q)(u; v) = (p[1, s], u[s+1, t]; v[t, 1]) = (p'[1, s], u[s+1, t]; v[t, 1]) = (p'; q')(u; v).$$

As $(u; v)$ is arbitrary, we conclude that $\lambda_{(p; q)} = \lambda_{(p'; q')}$ holds over D_t . In a dual manner, $(p; q) \mathcal{L}(p'; q')$ implies $\rho_{(p; q)} = \rho_{(p'; q')}$ over D_t . This suffices to confirm that $B(r_1, \dots, r_n)$ is a regular band. \square

Of course, we are going to choose the minimal \mathcal{D} -class, D_n , to be the one to which we associate the computed maximal subgroup of $\text{IG}(B(r_1, \dots, r_n))$. It already suggests a natural choice for the sets I and J , namely, I will be the set of all sequences $p[1, n]$ such that $p_m \in \{0, 1\}$ for all $1 \leq m \leq n$ (so that $|I| = 2^n$), while J will be the set of all sequences $q[n, 1]$ with $0 \leq q_m \leq r_m$ for each $1 \leq m \leq n$, implying $|J| = \prod_{m=1}^n (r_m + 1)$. The sequence of n zeros $\mathbf{0} = (0, \dots, 0)$ will be our highlighted element 0 from Theorem 2 and $e = (\mathbf{0}; \mathbf{0})$.

The next lemma describes the kernels and images of λ - and ρ -functions of elements of $B(r_1, \dots, r_n)$ belonging to the \mathcal{D} -class D_s .

Lemma 7. Let $1 \leq s \leq n$ and $\mathbf{a} = (u[1, s]; v[s, 1]) \in B(r_1, \dots, r_n)$. Then:

- (1) $(p'[1, n], p''[1, n]) \in \text{Ker } \lambda_{\mathbf{a}}$ if and only if $p'[s+1, n] = p''[s+1, n]$;
- (2) $(q'[n, 1], q''[n, 1]) \in \text{Ker } \rho_{\mathbf{a}}$ if and only if $q'[n, s+1] = q''[n, s+1]$;
- (3) $\text{Im } \lambda_{\mathbf{a}} = \{(u[1, s], p'_{s+1}, \dots, p'_n) : p'_m \in \{0, 1\} \text{ for all } s+1 \leq m \leq n\}$;
- (4) $\text{Im } \rho_{\mathbf{a}} = \{(q'_n, \dots, q'_{s+1}, v[s, 1]) : 0 \leq q'_m \leq r_m \text{ for all } s+1 \leq m \leq n\}$.

Proof. We prove only (1) and (3), while (2) and (4) are completely dual, and may be left to the reader as an exercise.

(1) $(p'[1, n], p''[1, n]) \in \text{Ker } \lambda_{\mathbf{a}}$ holds if and only if $(u; v)(p'; q) = (u; v)(p''; q)$ holds for arbitrary $q \in J$. Since $s \leq n$, the latter condition is equivalent to

$$(u[1, s], p'[s+1, n]) = (u[1, s], p''[s+1, n]),$$

and this is obviously equivalent to $p'[s+1, n] = p''[s+1, n]$.

(3) A (binary) sequence belongs to $\text{Im } \lambda_{\mathbf{a}}$ if and only if it is equal to the left component of the pair resulting from the product $(u; v)(p'; q')$ for some $(p'; q') \in D_n$. By definition, this component is equal to $(u[1, s], p'[s+1, n]) = (u[1, s], p'_{s+1}, \dots, p'_n)$; this completes the proof. \square

It transpires from the previous lemma that both the left-right and the up-down singularisation zones in D_n induced by a fixed \mathcal{D} -class D_s , $1 \leq s \leq n$, cover the entire grid $I \times J$; in fact, each of them defines a tiling of the grid, as each entry belongs to precisely one left-right zone and one up-down zone.

We have now set the stage to take on the main part of the proof of Theorem 1.

4. PROOF OF THEOREM 1, AND SEVERAL OPEN PROBLEMS

We set out with the presentation described in Theorem 2 and made precise by Lemmata 3 and 7. For $1 \leq s \leq n$ let \mathbf{u}_s be the binary sequence whose sole non-zero entry (equal to 1) occurs at position s . Furthermore, for $1 \leq r \leq r_s$ let $\mathbf{v}_s(r)$ be the sequence all of whose entries are equal to 0 except the one at position s from the right, which is equal to r . The general guiding idea of our proof is to eliminate as many generators and relations as possible, until we are left only with the generators

$$\xi_r^{(s)} = f_{\mathbf{u}_s, \mathbf{v}_s(r)}$$

and commutation relations

$$\xi_r^{(s)} \xi_{r'}^{(s')} = \xi_{r'}^{(s')} \xi_r^{(s)}$$

for all pairs of distinct indices s, s' , $1 \leq r \leq r_s$, $1 \leq r' \leq r_{s'}$. This is clearly a presentation of the required group G .

To achieve this, one of the key notions will be the *support* $\text{supp}(p; q)$ of the pair of sequences $(p; q)$, both of length n , as the set of all indices $s \in \{1, \dots, n\}$ for which both entries p_s and q_s are nonzero. It will turn out eventually that $f_{p,q}$ depends only on $\text{supp}(p; q)$ and the entries of q placed at positions given by $\text{supp}(p; q)$. The following is one of the main steps towards our goal, describing a crucial part of the topological structure of the complex induced by singular squares in D_n .

Lemma 8. *Let $(p; q) \in D_n$ be such that $\text{supp}(p; q) = \emptyset$. Then $f_{p,q} = 1$ is a consequence of \mathfrak{R} .*

Proof. We are going to show that $(p_0, q_0) = (\mathbf{0}, \mathbf{0})$ and (p, q) can be connected by a sequence of left-right and up-down singular squares, such that (p_0, q_0) is the ‘top-left’ corner of the first square, (p, q) is the ‘bottom-right’ corner of the last square, and any two adjacent squares share a side (that is, a pair of elements of $I \times J$ situated in the same column or in the same row).

To this end, we start by constructing a decreasing sequence

$$\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \dots$$

of indices from $\{1, \dots, n\}$. First of all, let α_1 be the position of the rightmost occurrence of 1 in p . If such occurrence does not exist, then we already have $p = \mathbf{0}$ and the lemma follows since $f_{\mathbf{0},q} = 1$ is contained in \mathfrak{R} . Next, suppose that the initial segment of our sequence $\alpha_1 > \dots > \alpha_k$ has been constructed for some $k \geq 1$. Let β_k be the largest index $\beta_k < \alpha_k$ such that $q_{\beta_k} \neq 0$ (in other words, the position of the leftmost nonzero entry in q prior to position α_k – remember that q is enumerated in reverse manner); if such index does not exist, then our sequence terminates. On the other hand, if the initial segment $\alpha_1 > \dots > \alpha_k > \beta_k$ has already been constructed for some $k \geq 1$, then we choose α_{k+1} to be the largest index $\alpha_{k+1} < \beta_k$ such that $p_{\alpha_{k+1}} = 1$; again, if such does not exist, then our construction terminates.

Now we define, for $k \geq 1$,

$$\begin{aligned} \mathbf{p}_k &= (0, \dots, 0, p[\beta_k + 1, n]), \\ \mathbf{q}_k &= (q[n, \alpha_k + 1], 0, \dots, 0), \end{aligned}$$

where in the definition of \mathbf{p}_k there are β_k zeros on the left, and α_k zeros on the right in \mathbf{q}_k . We claim that for any (available) k we have that

$$(\mathbf{p}_{k-1}, \mathbf{p}_k; \mathbf{q}_{k-1}, \mathbf{q}_k)$$

is an up-down singular square in D_n , while

$$(\mathbf{p}_{k-1}, \mathbf{p}_k; \mathbf{q}_k, \mathbf{q}_{k+1})$$

is a left-right singular square in D_n . The last step, connecting the last of these squares to (p, q) will be explained later, and there will be two cases depending on whether our sequence of indices ends with α_m or with β_m for some m .

To see this, we use Lemma 7. Let

$$\mathbf{a}_k = (\underbrace{0, \dots, 0}_{\alpha_k}; \underbrace{0, \dots, 0}_{\alpha_k}) \in D_{\alpha_k}.$$

We have $\mathbf{p}_{k-1}[\alpha_k + 1, n] = (0, \dots, 0, p[\beta_{k-1} + 1, n])$ since $\beta_{k-1} > \alpha_k$. On the other hand, $\mathbf{p}_k[\alpha_k + 1, n] = p[\alpha_k + 1, n]$, as $\alpha_k > \beta_k$; however, by the very definition of α_k we must have $p_{\alpha_k+1} = \dots = p_{\beta_k} = 0$, so that $\mathbf{p}_{k-1}[\alpha_k + 1, n] = \mathbf{p}_k[\alpha_k + 1, n]$. Therefore, $(\mathbf{p}_{k-1}, \mathbf{p}_k) \in \text{Ker } \lambda_{\mathbf{a}_k}$. Furthermore, $\mathbf{q}_{k-1}[\alpha_k, 1] = (0, \dots, 0) = \mathbf{q}_k[\alpha_k, 1]$ since $\alpha_{k-1} > \alpha_k$ (for convenience, we may add $\alpha_0 = \beta_0 = n$ to the definition), implying $\mathbf{q}_{k-1}, \mathbf{q}_k \in \text{Im } \rho_{\mathbf{a}_k}$. Hence, \mathbf{a}_k up-down singularises $(\mathbf{p}_{k-1}, \mathbf{p}_k; \mathbf{q}_{k-1}, \mathbf{q}_k)$. One shows in a similar fashion that the element

$$\mathbf{b}_k = (\underbrace{0, \dots, 0}_{\beta_k}; \underbrace{0, \dots, 0}_{\beta_k}) \in D_{\beta_k}$$

left-right singularises the square $(\mathbf{p}_{k-1}, \mathbf{p}_k; \mathbf{q}_k, \mathbf{q}_{k+1})$.

We now claim that \mathfrak{R} implies $f_{\mathbf{p}_k, \mathbf{q}_k} = 1$ for each available k (that is, for which both \mathbf{p}_k and \mathbf{q}_k exist). Indeed, this is immediately clear for $k = 1$, since $(\mathbf{0}, \mathbf{p}_1; \mathbf{0}, \mathbf{q}_1)$ is a singular square, so the relation $f_{\mathbf{0}, \mathbf{0}}^{-1} f_{\mathbf{0}, \mathbf{q}_1} = f_{\mathbf{p}_1, \mathbf{0}}^{-1} f_{\mathbf{p}_1, \mathbf{q}_1}$ belongs to \mathfrak{R} , as well as the relations $f_{\mathbf{0}, \mathbf{0}} = f_{\mathbf{0}, \mathbf{q}_1} = f_{\mathbf{p}_1, \mathbf{0}} = 1$. Proceeding by induction, assume that $f_{\mathbf{p}_k, \mathbf{q}_k} = 1$ has already been deduced from \mathfrak{R} and that the index $k+1$ is available as well. Then $(\mathbf{p}_k, \mathbf{p}_{k+1}) \in \text{Ker } \lambda_{\mathbf{a}_{k+1}}$ and $\mathbf{q}_k \in \text{Im } \rho_{\mathbf{a}_k}$. Notice, however, that $\text{Im } \rho_{\mathbf{a}_k} \subseteq \text{Im } \rho_{\mathbf{a}_{k+1}}$, yielding that the square $(\mathbf{p}_k, \mathbf{p}_{k+1}; \mathbf{0}, \mathbf{q}_k)$ is (up-down) singularised by \mathbf{a}_{k+1} . This implies $f_{\mathbf{p}_{k+1}, \mathbf{q}_k} = 1$ as all the other three generators corresponding to the vertices of the latter square are already proved to be equal to 1. Analogously, \mathbf{b}_{k+1} (left-right) singularises the square $(\mathbf{0}, \mathbf{p}_k; \mathbf{q}_k, \mathbf{q}_{k+1})$, resulting in deduction of the relation $f_{\mathbf{p}_k, \mathbf{q}_{k+1}} = 1$. Finally, since we already know that the square $(\mathbf{p}_k, \mathbf{p}_{k+1}; \mathbf{q}_k, \mathbf{q}_{k+1})$ is singular and thus

$$f_{\mathbf{p}_k, \mathbf{q}_k}^{-1} f_{\mathbf{p}_k, \mathbf{q}_{k+1}} = f_{\mathbf{p}_{k+1}, \mathbf{q}_k}^{-1} f_{\mathbf{p}_{k+1}, \mathbf{q}_{k+1}}$$

belongs to \mathfrak{R} , we obtain the desired relation $f_{\mathbf{p}_{k+1}, \mathbf{q}_{k+1}} = 1$.

It remains to perform the last step, connecting (p, q) to the previous chain of singular squares. As announced, we have two cases to consider. First, assume that our decreasing chain of indices looks as follows:

$$\alpha_1 > \beta_1 > \dots > \beta_{m-1} > \alpha_m.$$

This means that $q_\beta = 0$ for all $\beta < \alpha_m$, so that in fact $\mathbf{q}_m = q$. Now we show that $(\mathbf{p}_{m-1}, p; \mathbf{q}_{m-1}, q)$ is a singular square. Recall that

$$\mathbf{p}_{m-1} = (0, \dots, 0, p[\beta_{m-1} + 1, n]),$$

while

$$p = (p[1, \alpha_m], 0, \dots, 0, p[\beta_{m-1} + 1, n]),$$

whence $(\mathbf{p}_{m-1}, p) \in \text{Ker } \lambda_{\mathbf{a}_m}$. Also,

$$\mathbf{q}_{m-1} = (q[n, \alpha_{m-1} + 1], 0, \dots, 0),$$

$$q = (q[n, \alpha_m + 1], 0, \dots, 0),$$

thus $\mathbf{q}_{m-1}, q \in \text{Im } \rho_{\mathbf{a}_m}$, since $\alpha_m < \alpha_{m-1}$. It follows that \mathbf{a}_m up-down singularises the square $(\mathbf{p}_{m-1}, p; \mathbf{q}_{m-1}, q)$. This is also true for $(\mathbf{p}_{m-1}, p; \mathbf{0}, \mathbf{q}_{m-1})$. On the other hand, note that the square $(\mathbf{p}_{m-2}, \mathbf{p}_{m-1}; \mathbf{q}_{m-1}, \mathbf{q}_m)$ is left-right singularised by \mathbf{b}_{m-1} , implying $(\mathbf{q}_{m-1}, \mathbf{q}_m) \in \text{Ker } \rho_{\mathbf{b}_{m-1}}$. As both $\mathbf{0}$ and \mathbf{p}_{m-1} start with β_{m-1} zeros, we have $\mathbf{0}, \mathbf{p}_{m-1} \in \text{Im } \lambda_{\mathbf{b}_{m-1}}$; consequently, $(\mathbf{0}, \mathbf{p}_{m-1}; \mathbf{q}_{m-1}, q)$ is left-right singularised by \mathbf{b}_{m-1} . So, $f_{\mathbf{p}_{m-1}, q} = f_{p, \mathbf{q}_{m-1}} = 1$ follows from \mathfrak{R} , which, in the light of the fact that

$(\mathbf{p}_{m-1}, p; \mathbf{q}_{m-1}, q)$ is singularised, further implies $f_{p,q} = 1$, as required. The other case, when our chain of indices is of the form

$$\alpha_1 > \beta_1 > \cdots > \alpha_m > \beta_m$$

is, in a sense, dual. Namely, now we have $p_\alpha = 0$ for all $\alpha < \beta_m$, so $\mathbf{p}_m = p$. Similarly as above, it turns out that $(\mathbf{p}_{m-1}, p; \mathbf{q}_m, q)$ is left-right singularised by \mathbf{b}_m , as well as $(\mathbf{0}, \mathbf{p}_{m-1}; \mathbf{q}_m, q)$, while $(\mathbf{p}_{m-1}, p; \mathbf{0}, \mathbf{q}_m)$ is up-down singularised by \mathbf{a}_m . This renders $f_{\mathbf{p}_{m-1}, q} = f_{p, \mathbf{q}_m} = 1$ as consequences of \mathfrak{R} , with the same conclusion $f_{p,q} = 1$ as above. \square

The generators from \mathfrak{X} we have found in the previous lemma to be equal to 1 will act as a ‘basic frame’ for exploiting singular squares in order to find further pairs of generators that can be proved equal from \mathfrak{R} . This is precisely the content of the following lemma, for which the previous one is the initial (and, as it will turn out, the hardest) step.

Lemma 9. *Let $(p; q) \in D_n$ be such that $\text{supp}(p; q) = \{s_1, \dots, s_t\} \neq \emptyset$, where $s_1 < \dots < s_t$. Furthermore, let $\mathbf{u}(\bar{s})$ be the binary string in which 1’s occur exactly at positions $\bar{s} = (s_1, \dots, s_t)$, while $\mathbf{v}(\bar{s}, q)$ is the sequence having q_{s_i} at position s_i , $1 \leq i \leq t$, and 0 elsewhere (that is, $\mathbf{v}(\bar{s}, q)$ is obtained from q by turning each entry that is not at a position from $\text{supp}(p; q)$ to 0). Then*

$$f_{p,q} = f_{\mathbf{u}(\bar{s}), \mathbf{v}(\bar{s}, q)}$$

is a consequence of \mathfrak{R} . In particular, if $t = 1$, that is, if $\text{supp}(p; q) = \{s\}$, then \mathfrak{R} implies $f_{p,q} = \xi_{q_s}^{(s)}$.

Proof. We induct on $|\text{supp}(p; q)|$, the base of induction being established by the previous lemma. Assume that the statement of the lemma has been proved for all pairs of sequences p', q' such that $|\text{supp}(p'; q')| < t$ and consider two sequences p, q such that $|\text{supp}(p; q)| = t$. For all $1 \leq i \leq t$ define

$$p^{(i)} = (p[1, s_{t-i+1}], \bar{p}_i),$$

where \bar{p}_i is the binary string of length $n - s_{t-i+1}$ obtained from $p[s_{t-i+1} + 1, n]$ by turning to 0 all 1’s that are not at one of the positions s_{t-i+2}, \dots, s_t . Furthermore, let $q^{(i)}$ be the sequence obtained from q by turning its nonzero entry at position s_{t-i+1} into 0. Finally, let $\bar{q} = (0, \dots, 0, q[s_1 - 1, 1])$.

Now by Lemma 7 it follows that the element $(p[1, s_t]; 0, \dots, 0) \in D_{s_t}$ left-right singularises the square $(p, p^{(1)}; q, q^{(1)})$, while $(p[1, s_{t-i}]; 0, \dots, 0) \in D_{s_{t-i}}$ left-right singularises $(p^{(i)}, p^{(i+1)}; q, q^{(i+1)})$ for all $1 \leq i < t$. Since

$$\text{supp}(p; q^{(1)}) = \text{supp}(p^{(1)}; q^{(1)}) = \{s_1, \dots, s_{t-1}\}$$

and

$$\text{supp}(p^{(i)}; q^{(i+1)}) = \text{supp}(p^{(i+1)}; q^{(i+1)}) = \text{supp}(p; q) \setminus \{s_{t-i}\}$$

for all $i < t$, by the induction hypothesis we have that the relations

$$f_{p,q^{(1)}} = f_{p^{(1)}, q^{(1)}} \quad \text{and} \quad f_{p^{(i)}, q^{(i+1)}} = f_{p^{(i+1)}, q^{(i+1)}}$$

follow from \mathfrak{R} . Bearing in mind the considered singular squares, this implies that

$$f_{p,q} = f_{p^{(1)}, q} = \cdots = f_{p^{(t)}, q}$$

can be deduced from \mathfrak{R} . Finally, the square $(p^{(t)}, \mathbf{u}(\bar{s}); q, \bar{q})$ is up-down singularised by $(0, \dots, 0; q[s_1 - 1, 1]) \in D_{s_1 - 1}$ and $\text{supp}(p^{(t)}; \bar{q}) = \text{supp}(\mathbf{u}(\bar{s}); \bar{q}) = \emptyset$, so $f_{p^{(t)}, \bar{q}} = f_{\mathbf{u}(\bar{s}), \bar{q}} = 1$ follows from \mathfrak{R} by the previous lemma. Hence, we deduce $f_{p^{(t)}, q} = f_{\mathbf{u}(\bar{s}), q}$; altogether, we obtain

$$f_{p,q} = f_{\mathbf{u}(\bar{s}), q}.$$

A completely dual argument to the one presented in the previous paragraph would show that

$$f_{\mathbf{u}(\bar{s}), q} = f_{\mathbf{u}(\bar{s}), \mathbf{v}(\bar{s}, q)}$$

is a consequence of \mathfrak{R} , thus finishing the inductive proof. \square

The only remaining major task is to relate the generators of the form $f_{\mathbf{u}(\bar{s}), \mathbf{v}(\bar{s}, q)}$ (all others are already proved to be redundant) to the ‘renamed’ generators $\xi_r^{(s)}$ and to deduce the corresponding commutation relations.

Lemma 10. *Let $q \in J$ be arbitrary and $\bar{s} = (s_1, \dots, s_t)$, with all the further notation and conventions as in the previous lemma. Then \mathfrak{R} implies the relation*

$$f_{\mathbf{u}(\bar{s}), \mathbf{v}(\bar{s}, q)} = \xi_{q_{s_1}}^{(s_1)} \dots \xi_{q_{s_t}}^{(s_t)}, \quad (4.1)$$

as well as each relation of the form

$$\xi_r^{(s)} \xi_{r'}^{(s')} = \xi_{r'}^{(s')} \xi_r^{(s)} \quad (4.2)$$

for all $s \neq s'$, $1 \leq r \leq r_s$ and $1 \leq r' \leq r_{s'}$.

Proof. Since the case $t = 1$ already follows from the previous lemma, we assume that $t \geq 2$. We first consider the case $t = 2$, establishing the commutation relations (4.2) along the way, and then proceed by induction on t .

Let $s < s'$ and $\bar{s} = (s, s')$. Consider the square $(\mathbf{u}_{s'}, \mathbf{u}(s, s'); \mathbf{v}_s(q_s), \mathbf{v}((s, s'), q))$; since $\mathbf{u}_{s'}[s+1, n] = \mathbf{u}(s, s')[s+1, n]$ and $\mathbf{v}_s(q_s)[s, 1] = \mathbf{v}((s, s'), q)[s, 1]$ it follows that this square is up-down singularised by $(0, \dots, 0; q_s, 0, \dots, 0) \in D_s$. As $\text{supp}(\mathbf{u}_{s'}; \mathbf{v}_s(q_s)) = \emptyset$, $\text{supp}(\mathbf{u}_{s'}; \mathbf{v}((s, s'), q)) = \{s'\}$ and $\text{supp}(\mathbf{u}(s, s'); \mathbf{v}_s(q_s)) = \{s\}$, the previous two lemmata allow us to obtain the relations

$$\begin{aligned} f_{\mathbf{u}_{s'}, \mathbf{v}_s(q_s)} &= 1, \\ f_{\mathbf{u}_{s'}, \mathbf{v}((s, s'), q)} &= \xi_{q_{s'}}^{(s')}, \\ f_{\mathbf{u}(s, s'), \mathbf{v}_s(q_s)} &= \xi_{q_s}^{(s)}. \end{aligned}$$

Therefore, Theorem 2 and Lemma 3 provide us with the relation

$$f_{\mathbf{u}_{s'}, \mathbf{v}_s(q_s)}^{-1} f_{\mathbf{u}_{s'}, \mathbf{v}((s, s'), q)} = f_{\mathbf{u}(s, s'), \mathbf{v}_s(q_s)}^{-1} f_{\mathbf{u}(s, s'), \mathbf{v}((s, s'), q)},$$

which yields

$$f_{\mathbf{u}(s, s'), \mathbf{v}((s, s'), q)} = \xi_{q_s}^{(s)} \xi_{q_{s'}}^{(s')}.$$

On the other hand, the square $(\mathbf{u}_s, \mathbf{u}(s, s'); \mathbf{v}_{s'}(q_{s'}), \mathbf{v}((s, s'), q))$ is left-right singularised by $(0, \dots, 0; 1; 0, \dots, 0) \in D_s$. Since $\text{supp}(\mathbf{u}_s; \mathbf{v}_{s'}(q_{s'})) = \emptyset$, $\text{supp}(\mathbf{u}_s; \mathbf{v}((s, s'), q)) = \{s\}$ and $\text{supp}(\mathbf{u}(s, s'); \mathbf{v}_{s'}(q_{s'})) = \{s'\}$, for the first three generators in the relation

$$f_{\mathbf{u}_s, \mathbf{v}_{s'}(q_{s'})}^{-1} f_{\mathbf{u}_s, \mathbf{v}((s, s'), q)} = f_{\mathbf{u}(s, s'), \mathbf{v}_{s'}(q_{s'})}^{-1} f_{\mathbf{u}(s, s'), \mathbf{v}((s, s'), q)}$$

we obtain

$$\begin{aligned} f_{\mathbf{u}_s, \mathbf{v}_{s'}(q_{s'})} &= 1, \\ f_{\mathbf{u}_s, \mathbf{v}((s, s'), q)} &= \xi_{q_s}^{(s)}, \\ f_{\mathbf{u}(s, s'), \mathbf{v}_{s'}(q_{s'})} &= \xi_{q_{s'}}^{(s')}, \end{aligned}$$

yielding

$$f_{\mathbf{u}(s, s'), \mathbf{v}((s, s'), q)} = \xi_{q_{s'}}^{(s')} \xi_{q_s}^{(s)}.$$

Hence, we get

$$\xi_{q_s}^{(s)} \xi_{q_{s'}}^{(s')} = \xi_{q_{s'}}^{(s')} \xi_{q_s}^{(s)}$$

as a consequence of \mathfrak{R} . As $q \in J$ is arbitrary, it follows that all the commutation relations (4.2) are deduced, and that the first part of the lemma is verified for $t = 2$.

The remaining part of the proof follows by induction on t . Consider the sequence of indices $\bar{s} = (s_1, \dots, s_t)$ while assuming that the lemma holds for all shorter sequences. Let $\mathbf{u}'(\bar{s})$ be the binary sequence obtained from $\mathbf{u}(\bar{s})$ by turning its entry $q_{s_1} = 1$ at position

s_1 into 0. We note that the square $(\mathbf{u}'(\bar{s}), \mathbf{u}(\bar{s}); \mathbf{v}_{s_1}(q_{s_1}), \mathbf{v}(\bar{s}, q))$ is up-down singularised by $(0, \dots, 0; 1, 0, \dots, 0) \in D_{s_1}$. Furthermore, we have

$$\begin{aligned}\text{supp}(\mathbf{u}'(\bar{s}); \mathbf{v}_{s_1}(q_{s_1})) &= \emptyset, \\ \text{supp}(\mathbf{u}'(\bar{s}); \mathbf{v}(\bar{s}, q)) &= \{s_2, \dots, s_t\}, \\ \text{supp}(\mathbf{u}(\bar{s}); \mathbf{v}_{s_1}(q_{s_1})) &= \{s_1\},\end{aligned}$$

thus by the induction hypothesis and the previous lemmata \mathfrak{R} implies

$$\begin{aligned}f_{\mathbf{u}'(\bar{s}), \mathbf{v}_{s_1}(q_{s_1})} &= 1, \\ f_{\mathbf{u}'(\bar{s}), \mathbf{v}(\bar{s}, q)} &= \xi_{q_{s_2}}^{(s_2)} \dots \xi_{q_{s_t}}^{(s_t)}, \\ f_{\mathbf{u}(\bar{s}), \mathbf{v}_{s_1}(q_{s_1})} &= \xi_{q_{s_1}}^{(s_1)}.\end{aligned}$$

The considered singular square yields the relation

$$f_{\mathbf{u}'(\bar{s}), \mathbf{v}_{s_1}(q_{s_1})}^{-1} f_{\mathbf{u}'(\bar{s}), \mathbf{v}(\bar{s}, q)} = f_{\mathbf{u}(\bar{s}), \mathbf{v}_{s_1}(q_{s_1})}^{-1} f_{\mathbf{u}(\bar{s}), \mathbf{v}(\bar{s}, q)},$$

implying

$$f_{\mathbf{u}(\bar{s}), \mathbf{v}(\bar{s}, q)} = \xi_{q_{s_1}}^{(s_1)} \xi_{q_{s_2}}^{(s_2)} \dots \xi_{q_{s_t}}^{(s_t)},$$

as wanted. \square

Proof of Theorem 1. The combined effect of the previous three lemmata is that any generator $f_{p,q} \in \mathfrak{X}$ can be expressed in terms of $\xi_r^{(s)}$ such that $1 \leq s \leq n$, $1 \leq r \leq r_s$; in addition, $\xi_r^{(s)}$ and $\xi_{r'}^{(s')}$ commute whenever $s \neq s'$. In turn, we are now going to prove that each relation from \mathfrak{R} is implied by (4.1) and (4.2). Indeed, let $(p, p'; q, q')$ be a singular square in D_n ; for the sake of an example, assume it is of the left-right type (the other case is dual). Then for some s we have $p[1, s] = p'[1, s]$ and $q[n, s+1] = q'[n, s+1]$. This means that an index $\sigma > s$ belongs to $\text{supp}(p; q)$ ($\text{supp}(p'; q)$) if and only if it belongs to $\text{supp}(p; q')$ (resp. $\text{supp}(p'; q')$), so that $\text{supp}(p; q)$ and $\text{supp}(p; q')$ (resp. $\text{supp}(p'; q)$ and $\text{supp}(p'; q')$) differ only in their members not exceeding s . But, on the other hand, we also have that an index $\tau \leq s$ belongs to $\text{supp}(p; q)$ ($\text{supp}(p; q')$) if and only if it belongs to $\text{supp}(p'; q)$ (resp. $\text{supp}(p'; q')$). Therefore, there are four sets of indices $A, B \subseteq \{1, \dots, s\}$ and $X, Y \subseteq \{s+1, \dots, n\}$ such that

$$\begin{aligned}\text{supp}(p; q) &= A \cup X, \quad \text{supp}(p; q') = B \cup X, \\ \text{supp}(p'; q) &= A \cup Y, \quad \text{supp}(p'; q') = B \cup Y.\end{aligned}$$

This and (4.1) provides us with sufficient information to establish that

$$\begin{aligned}f_{p,q} &= \prod_{\sigma \in A} \xi_{q_\sigma}^{(\sigma)} \cdot \prod_{\tau \in X} \xi_{q_\tau}^{(\tau)}, \\ f_{p,q'} &= \prod_{\sigma \in B} \xi_{q'_\sigma}^{(\sigma)} \cdot \prod_{\tau \in X} \xi_{q'_\tau}^{(\tau)} = \prod_{\sigma \in B} \xi_{q'_\sigma}^{(\sigma)} \cdot \prod_{\tau \in X} \xi_{q_\tau}^{(\tau)}, \\ f_{p',q} &= \prod_{\sigma \in A} \xi_{q_\sigma}^{(\sigma)} \cdot \prod_{\tau \in Y} \xi_{q_\tau}^{(\tau)}, \\ f_{p',q'} &= \prod_{\sigma \in B} \xi_{q'_\sigma}^{(\sigma)} \cdot \prod_{\tau \in Y} \xi_{q'_\tau}^{(\tau)} = \prod_{\sigma \in B} \xi_{q'_\sigma}^{(\sigma)} \cdot \prod_{\tau \in Y} \xi_{q_\tau}^{(\tau)}.\end{aligned}$$

Given the commutation relations (4.2), we instantly get

$$f_{p,q}^{-1} f_{p,q'} = \left(\prod_{\sigma \in A} \xi_{q_\sigma}^{(\sigma)} \right)^{-1} \cdot \prod_{\sigma \in B} \xi_{q'_\sigma}^{(\sigma)} = f_{p',q}^{-1} f_{p',q'}.$$

At this point, we have all the necessary ingredients to claim that the presentation $\langle \mathfrak{X} | \mathfrak{R} \rangle$ can be transformed, by Tietze transformations, into $\langle \mathfrak{X}' | \mathfrak{C} \rangle$, where $\mathfrak{X}' = \{\xi_r^{(s)} : 1 \leq s \leq n, 1 \leq r \leq r_s\}$ and \mathfrak{C} is the set of all commutation relations (4.2). However, the

latter transformation obviously defines the direct product $G = F_{r_1} \times \cdots \times F_{r_n}$, so we are done. \square

Corollary 11. *The maximal subgroup of $\text{IG}(B(1, \dots, 1))$ (n times 1) corresponding to the \mathcal{D} -class D_s of $B(1, \dots, 1)$, $1 \leq s \leq n$, is the free abelian group of rank s . Consequently, for any idempotent generated semigroup S that has the same biordered set of idempotents as $B(1, \dots, 1)$, all maximal subgroups of $\text{IG}(S)$ are abelian.*

Remark 12. The regular band B constructed in [6, Proposition 3] is just $B(1, 1)$. Hence, this proposition follows immediately from the previous corollary.

In view of our Theorem 1, the depiction of the singularisation zones at the end of Sect. 2, and Lemma 7, it is fairly obvious that there is considerable room for generalisation to other torsion-free groups, which may appear as fundamental groups of (Graham-Houghton) complexes of bands, and even the simplest case of regular bands offers possibilities galore for further work. For example, it is not too difficult to see that the methods of this note can be used to obtain groups of the form

$$\dots (((((F_{r_1} \times F_{r_2}) * F_{r_3}) \times F_{r_4}) * F_{r_5}) \times \dots,$$

where $*$ denotes the free product of groups (at least for some particular values of ranks r_3, r_5, \dots), as maximal subgroups of $\text{IG}(B)$ for a suitably chosen regular band B . So, the following arises rather naturally.

Question 1. Which groups assembled from infinite cyclic groups \mathbb{Z} by means of direct and free products arise as maximal subgroups of $\text{IG}(B)$ for a (regular) band B ?

Droms et al. [9] remark that the groups described in the previous question are special cases of *right-angled Artin groups* [4] (that is, graph groups), whose underlying graphs omit the path of length 3 as an induced subgraph (on any four vertices). Thus we may pose a slightly broader question.

Question 2. Which right-angled Artin groups arise as maximal subgroups of $\text{IG}(B)$ for a (regular) band B ?

Also, the following seems both interesting and, at this point, feasible to conjecture.

Question 3. Is it true for each (regular) band B that the maximal subgroups of $\text{IG}(B)$ must be torsion-free?

REFERENCES

- [1] G. Baumslag and J. E. Roseblade, Subgroups of direct products of free groups, *J. London Math. Soc.* (2) **30** (1984), 44–52.
- [2] M. Bridson, J. Howie, C. F. Miller, III and H. Short, The subgroups of direct products of surface groups, *Geom. Dedicata* **92** (2002), 95–103.
- [3] M. Brittenham, S. W. Margolis and J. Meakin, Subgroups of free idempotent generated semigroups need not be free, *J. Algebra* **321** (2009), 3026–3042.
- [4] R. Charney, An introduction to right-angled Artin groups, *Geom. Dedicata* **125** (2007), 141–158.
- [5] W. Dison, A subgroup of a direct product of free groups whose Dehn function has a cubic lower bound, *J. Group Theory* **12** (2009), 783–793.
- [6] I. Dolinka, A note on maximal subgroups of free idempotent generated semigroups over bands, *Periodica Math. Hungar.*, to appear. [arXiv:1010.3737v3](https://arxiv.org/abs/1010.3737v3)
- [7] I. Dolinka, A note on free idempotent generated semigroups over the full monoid of partial transformations, *Comm. Algebra*, to appear. [arXiv:1101.3057v2](https://arxiv.org/abs/1101.3057v2)
- [8] I. Dolinka and R. D. Gray, Maximal subgroups of free idempotent generated semigroups over the full linear monoid, *Trans. Amer. Math. Soc.*, to appear. [arXiv:1112.0893](https://arxiv.org/abs/1112.0893)
- [9] C. Droms, B. Servatius and H. Servatius, Groups assembled from free and direct products, *Discrete Math.* **109** (1992), 69–75.
- [10] D. Easdown, Biordered sets of bands, *Semigroup Forum* **29** (1984), 241–246.
- [11] D. Easdown, Biordered sets are biordered subsets of idempotents of semigroups, *J. Austral. Math. Soc. Ser. A* **37** (1984), 258–268.

- [12] D. Easdown, Biordered sets come from semigroups, *J. Algebra* **96** (1985), 581–591.
- [13] M. Eliyahu, E. Liberman, M. Schaps and M. Teicher, The characterization of a line arrangement whose fundamental group of the complement is a direct sum of free groups, *Algebr. Geom. Topol.* **10** (2010), 1285–1304.
- [14] J. A. Erdos, On products of idempotent matrices, *Glasgow Math. J.* **8** (1967), 118–122.
- [15] K.-M. Fan, Direct product of free groups as the fundamental group of the complement of a union of lines, *Michigan Math. J.* **44** (1997), 283–291.
- [16] R. Gray and N. Ruškuc, On maximal subgroups of free idempotent generated semigroups, *Israel J. Math.*, to appear.
- [17] R. Gray and N. Ruškuc, Maximal subgroups of free idempotent generated semigroups over the full transformation monoid, *Proc. London Math. Soc.*, to appear. [arXiv:1101.1833](https://arxiv.org/abs/1101.1833)
- [18] P. M. Higgins, *Techniques of Semigroup Theory*, Oxford University Press, New York, 1992.
- [19] J. M. Howie, The subsemigroup generated by the idempotents of a full transformation semigroup, *J. London Math. Soc.* **41** (1966), 707–716.
- [20] J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford University Press, New York, 1995.
- [21] B. McElwee, Subgroups of the free semigroup on a biordered set in which principal ideals are singletons, *Comm. Algebra* **30** (2002), 5513–5519.
- [22] C. F. Miller, III, Subgroups of direct products with a free group, *Q. J. Math.* **53** (2002), 503–506.
- [23] K. S. S. Nambooripad, Structure of regular semigroups. I, *Mem. Amer. Math. Soc.* **22** (1979), no. 224, vii+119 pp.
- [24] K. S. S. Nambooripad and F. Pastijn, Subgroups of free idempotent generated regular semigroups, *Semigroup Forum* **21** (1980), 1–7.
- [25] N. Ruškuc, Presentations for subgroups of monoids, *J. Algebra* **220** (1999), 365–380.
- [26] F. Pastijn, The biorder on the partial groupoid of idempotents of a semigroup, *J. Algebra* **65** (1980), 147–187.
- [27] M. Petrich, *Lectures in Semigroups*, Wiley, New York, 1977.
- [28] H. Short, Finitely presented subgroups of a product of two free groups, *Q. J. Math.* **52** (2001), 127–131.
- [29] K. Williams, Line arrangements and direct products of free groups, *Algebr. Geom. Topol.* **11** (2011), 587–604.

DEPARTMENT OF MATHEMATICS AND INFORMATICS, UNIVERSITY OF NOVI SAD, TRG DOSITEJA OBRADOVIĆA 4, 21101 NOVI SAD, SERBIA

E-mail address: dockie@dmi.uns.ac.rs